Factorization Algebras Associated to the (2, 0) Theory III

Kevin Costello
Notes by Qiaochu Yuan

December 11, 2014

Last time we looked at three twists of 3d $N = 4$ Yang-Mills. There was a holomorphic twist and two topological twists, the A-twist

$$\mathrm{Maps}(\mathbb{C}_{\bar{\partial}} \times \mathbb{R}_{dR} \to (V//G)_{dR}) \tag{1}$$

and the B-twist (Rozansky-Witten theory)

$$\mathrm{Maps}(\mathbb{R}^3_{dR} \to V//G). \tag{2}$$

For 4d $N = 2$ there are again three twists. There is the holomorphic twist

$$\mathrm{Maps}(\mathbb{C}^{2|1}, V//G), \tag{3}$$

the Donaldson twist

$$\mathrm{Maps}(\mathbb{C}^2_{dR} \to (V//G)_{dR}) \tag{4}$$

and the Kapustin half-twist

$$\mathrm{Maps}(\mathbb{C}_{\bar{\partial}} \times \mathbb{R}^2_{dR} \to V//G). \tag{5}$$

Consider the dimensional reduction of this last theory on a circle. Maps $S^1_{dR} \to X$ are almost maps $\mathrm{Spec}\, H(S^1) \to X$, which are maps $\mathbb{C}^{0|1} \to X$. Working equivariantly gives us the A-twist.

Earlier we saw that line operators on 4d $N = 2$ pure gauge theory ($V = 0$) should be identified with equivariant coherent sheaves

$$\mathrm{Coh}^{G[[t]]}(\mathrm{Gr}) \tag{6}$$

on the affine Grassmannian. On the other hand, it turns out that Wilson lines should be identified with modules over the Yangian $Y(\mathfrak{g} \oplus \mathfrak{g}^*)$. So we conjectured that

$$\mathrm{Coh}_0^{G[[t]]}(\mathrm{Gr}) \cong \mathrm{Mod}(Y(\mathfrak{g} \oplus \mathfrak{g}^*)) \tag{7}$$

as monoidal categories, where the subscript 0 indicates coherent sheaves supported at the origin. The monoidal structure on line operators is given by the operator product expansion (OPE) in the topological $\mathbb{R}^2$, but there is also an OPE in the holomorphic $\mathbb{C}$. This ought to correspond to an R-matrix with spectral parameter on the Yangian side.

Q: what about the 't Hooft operators?

A: the Yangian comes from doing perturbation theory. Perturbation theory can't see the 't Hooft operator by default. You need to do perturbation theory around a 't Hooft operator, and instead of a Yangian you find the universal enveloping algebra of a certain subalgebra. This would be the 4d analogue of calculations of monopole operators in 3d.

Now consider 4d $N = 4$. This involves considering

$$\mathrm{Maps}(\mathbb{C}^{2|1}, (T^*\mathfrak{g})//G) \tag{8}$$

which, by a suitable adjunction, turns out to be

$$\mathrm{Maps}(\mathbb{C}^{2|3}, BG). \tag{9}$$

What should we do in 5d $N = 2$? We saw that any odd $\mathbb{C}$ has the potential to come from a de Rham circle, and that is indeed what happens here. There is a twist

$$\mathrm{Maps}(\mathbb{C}^{2|2}_{\bar\partial} \times \mathbb{R}_{dR}, BG). \tag{10}$$

The remaining supercharges act on this space of maps by some vector fields, namely $\frac{\partial}{\partial \varepsilon_i}, \varepsilon_i \frac{\partial}{\partial z_j}$. The remaining R-symmetry is SL_2 rotating the two odd directions.

There is a further A-twist given by taking $\frac{\partial}{\partial \varepsilon_i}$ which gives

$$\mathrm{Maps}(\mathbb{C}^{2|1} \times \mathbb{R}, BG)_{dR}. \tag{11}$$

This theory counts holomorphic / flat bundles on $\mathbb{C}^2 \times \mathbb{R}$. The Hilbert space of this theory will turn out to be the cohomology of instanton moduli space.

There is also a further B-twist given by taking

$$\varepsilon_1 \frac{\partial}{\partial z_1} + \varepsilon_2 \frac{\partial}{\partial z_2} \tag{12}$$

which gives

$$\mathrm{Maps}(\mathbb{C}^2_{dR}, \mathbb{R}_{dR}, BG) \tag{13}$$

and this corresponds to local systems on $\mathbb{R}^5$.

There is in fact a $\mathbb{CP}^1$ worth of twists given by taking

$$\lambda \frac{\partial}{\partial \varepsilon_1} + \mu \varepsilon_2 \frac{\partial}{\partial z_2}. \tag{14}$$

When $\mu = 0$ this is the A-twist. When $\lambda = 0$ this is a mixed holomorphic / topological twist

$$\mathrm{Maps}(\mathbb{C}^{1|1} \times \mathbb{R}^3, BG). \tag{15}$$

When $\lambda, \mu \neq 0$ this is an interesting topological twist.

What's the phase space of the A-twist? Let X be a complex surface with $c_1(X) = 0$ and let's consider

$$\mathrm{Maps}(X \times \mathbb{C}^{0|1} \times \mathbb{R}_{dR}, BG_{dR}). \tag{16}$$

This gives

$$\Pi T\,\mathrm{Bun}_G(X)_{dR} \cong T^*\,\mathrm{Bun}_G(X)_{dR} \tag{17}$$

and we get

$$Z(X) = H^\bullet(\mathrm{Bun}_G(X)). \tag{18}$$

Before the A-twist we get $Z(X) = \Omega^\bullet(\mathrm{Bun}_G(X))$, but the A-twist introduces the de Rham differential.

Q: what's the significance of the condition that $c_1(X) = 0$?
A: without it, we'd have to explicitly talk about the canonical bundle.

Hence we expect

$$Z(X \times S^1) = \chi(\mathrm{Bun}_G(X)). \tag{19}$$

Local operators in this theory should be an E_5 algebra acting on $H^\bullet(\mathrm{Bun}_G(X))$ via Hecke correspondences, since they change instantons at one point.

Conjecture 0.1. *Take $G = \mathbb{C}^\times$. Then local operators should be an E_5 version of the Grojnowski-Nakajima-Heisenberg algebra $\mathbb{C}[\alpha_n], n \in \mathbb{Z}$ acting on the cohomology of instanton moduli space, with Poisson bracket given by*

$$\{\alpha_n, \alpha_m\} = n\delta_{n+m,0}. \tag{20}$$

Equivalently, it is the E_5-enveloping algebra of the Heisenberg algebra. If we work equivariantly with respect to rotations of $\mathbb{C}^2$, we get an E_1 algebra which is the honest Heisenberg algebra acting on the equivariant cohomology of instanton moduli space.

What happens in the nonabelian case?

Conjecture 0.2. *Local operators should be an E_5 analogue of the affine W-algebra. If we work equivariantly with respect to rotations of $\mathbb{C}^2$, we get the W-algebra acting on the equivariant cohomology of instanton moduli space as conjectured by AGT, Maulik-Okounkov, Braverman-Finkelberg-Nakajima, Schiffman-Vasserot...*

What seems to be true is that the BPS part of SUSY just is geometric representation theory.

Earlier we specified that X was a complex surface with $c_1(X) = 0$ because we were being lazy. Really the ε appearing is dz, and more precisely we should write the relevant twist as

$$\mathrm{Maps}((\Pi T\mathbb{C}) \times \mathbb{R}^3_{dR}, BG). \tag{21}$$

When we replace $\mathbb{C}$ with a Riemann surface Σ we get

$$\mathrm{Maps}((\Pi T\Sigma) \times \mathbb{R}^3_{dR}, BG) \cong \mathrm{Maps}(\mathbb{R}^3_{dR}, \mathrm{Higgs}_G(\Sigma)) \tag{22}$$

and hence Rozansky-Witten theory on the Hitchin space. The full B-twist gives us Rozansky-Witten theory on $\mathrm{Loc}_G(\Sigma)$.

Q: just to be clear, when you write e.g. $\mathbb{R}^3_{dR}$ you're really specifying a 3d field theory on a point and allowing us to replace $\mathbb{R}^3$ with a 3-manifold or something later?

A: yes.

Let's discuss 5d gauge theory local operators in the A-twist based on the 3d monopole discussion. Replacing S^4 with a punctured cylinder as before, we expect to get the equivariant homology of the stack of pairs of torsion-free sheaves on $\mathbb{C}^2$ together with an isomorphism away from 0. This is some 2d version of the equivariant homology of the affine Grassmannian. The 2d version of the affine Grassmannian itself makes an appearance with Neumann boundary conditions (meaning that the bundle is trivial somewhere).